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New solutions of Yang–Baxter equations: Birman–Wenzl algebra and quantum group structures

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Abstract. New braid group representations associated with the fundamental representations of B_n, C_n and D_n are derived They are shown to satisfy Birman-Wenzl algebra By Yang-Baxterization we obtain new solutions of Yang-Baxter equations, including the twisted extensions Quantum group structures related to these new solutions are explicitly given

1. Introduction

The solutions of Yang–Baxter equations (YBE) attract more and more attention because of their importance in both mathematics and physics, such as statistical models [1]†, scattering matrices [2], knot theory [3]‡, conformal field theory (CFT) [4]§, quantum groups [5-9], and so on. Typically explicit examples of the solutions of YBE are those associated with the fundamental representations of B_n, C_n and D_n (as well as $A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$) as given by Jimbo [7], Reshetikhin [8] and others [10, 11]. It is well known that these solutions possess 'perfect' properties; for example, the usual classical limits, in accordance with the general conclusion of Belavin and Drinfeld [12], and the q-analogue CG coefficients [8, 13] and also the standard way of constructing the q-eigenvalues and the corresponding q-projectors [8] Moreover, this type of solution is connected with the current form of the quantum group [6-8]. We thus call them 'standard solutions' of YBE.

In our previous work we developed a systematic prescription to generate the solutions of YBE [14-16]. The basic idea is as follows. Firstly, we solve the braid relations to give the braid group representations (BGRs). This step obtains the asymptotic solutions of S-matrices (with infinity rapidity). In the process of the weight conservation should be taken into account. Secondly, for a given BGR we present a general prescription of Yang–Baxterization to generate the corresponding solution of YBE [15]. Finally,

† This reprinted volume collects together many original articles

‡ A general reference

§ Examples of the connection between braid group representations and CFT

0305-4470/91/122679+12$03 50

for given BGRs we follow Faddeev–Reshetikhin–Takhtajan (FRT) [17] to establish the related quantum group (QG) structures which, in general, may not be the same as the standard ones if the BGRs are not standard [18].

Therefore, we see that BGRs play the central role in finding solutions of YBE and constructing QG structures from the point of view of FRT.

In this paper we would like to present a general form of BGRs associated with the fundamental representations of B_n, D_n and C_n that after Yang–Baxterization automatically generate the twisted extensions; for example, $A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$ corresponding to B_n and D_n. This solution system contains the standard family as a particular case; however, there exists a new family of BGRs. We call these new solutions 'exotic solutions' to distinguish them from the standard ones. Actually, the exotic BGRs associated with $\mathrm{Sl}_q(2)$ was first introduced by Lee *et al* [18]

2. New solutions of BGRs

For completeness we first list the standard solutions of BGRs denoted by T for $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$, as given by Jimbo [7]:

$$T = q\sum_{k\neq 0} e_{kk}\otimes e_{kk} + w\sum_{\substack{k<m\\ k+m\neq 0}} e_{kk}\otimes e_{mm} + \sum_{\substack{k\neq m\\ k+m\neq 0}} e_{km}\otimes e_{mk}$$

$$+\sum_{k,m} a_{km} e_{k-m}\otimes e_{-km} \qquad (e_{km})_{ab} = \delta_{ka}\delta_{mb} \tag{2.1}$$

where $w = q - q^{-1}$ and

$$a_{km} = \begin{cases} 1 & (k=m=0) \\ q^{-1} & (k=m\neq 0) \\ w(\delta_{k-m} - \varepsilon_k\varepsilon_m q^{\bar{k}-\bar{m}}) & (k<m) \end{cases} \tag{2.2}$$

$\varepsilon_k = 1(-(2N-1)/2 \leqslant k \leqslant -\frac{1}{2})$, $\varepsilon_k = -1(\frac{1}{2} \leqslant k \leqslant (2N-1)/2)$ for $C_n^{(1)}$ and $\varepsilon_k = 1$ for $B_n^{(1)}$, $D_n^{(1)}$.

$$\bar{k} = \begin{cases} k+\frac{1}{2} & (-(N-1)/2 \leqslant k < 0) \\ k & (k=0) \\ k-\frac{1}{2} & (0<k\leqslant (2n-1)/2) \end{cases} \tag{2.3}$$

for $B_n^{(1)}$, $D_n^{(1)}$,

$$\bar{k} = \begin{cases} k-\frac{1}{2} & (-(2n-1)/2 \leqslant k \leqslant -\frac{1}{2}) \\ k+\frac{1}{2} & (\frac{1}{2} \leqslant k \leqslant (2n-1)/2) \end{cases} \tag{2.4}$$

for $C_n^{(1)}$.

The labelling set above is taken to be

$$L = [(N-1)/2, (N-1)/2-1, \ldots, -(N-1)/2] \tag{2.5}$$

which is a little different from that of Jimbo [7] $N = 2n+1$ for B_n and $N = 2n$ for D_n and C_n.

Now we extend the result to cover more general cases

By the weight conservation and CP invariance the BGR T has the general form

$$T=\sum_k u_k e_{kk}\otimes e_{kk}+\sum_{k<m} w_{k+m}^{(m)} e_{kk}\otimes e_{mm}+\sum_{k\neq m} p_{k+m}^{(k,m)} e_{km}\otimes e_{mk}$$
$$+\sum_{k,m} q^{(k,m)} e_{k-m}\otimes e_{-km} \qquad (k, m \in L\ (e_{km})_{ab}=\delta_{ka}\delta_{mb}) \tag{2.6}$$

where

$$p_{a+b}^{(a,b)}=p_{a+b}^{(b,a)} \qquad q^{(a,c)}=q^{(c,a)} \tag{2.7}$$

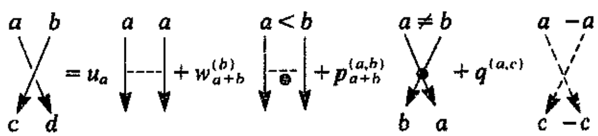
and

$$q^{(a,c)}\big|_{a=\pm c}=q^{(a,c)}\big|_{|a|<|c|,\ c>0}=q^{(a,0)}\big|_{a>0}=0 \tag{2.8}$$

in which

$$a, b, c, d\in[(N-1)/2, (N-2)/2-1, \ldots, -(N-1)/2]. \tag{2.9}$$

Equations (2.7) and (2.8) come from the restrictions of the 'six-vertex-type' solutions. Following our previous works [14] the corresponding diagrammatic expression reads



(2.10)

where

$=\begin{cases}1 & a=c, b=d, a<b\\ 0 & \text{otherwise}\end{cases}$

$=\begin{cases}1 & a=d\neq b=c\\ 0 & \text{otherwise.}\end{cases}$

Substituting (2.6) into the braid relations after tedious calculations, including the use of the extended diagrammatic technique [14], we derive the following general solutions of BGRs:

$$T=\sum_{k\neq 0} u_k e_{kk}\otimes e_{kk}+w\sum_{\substack{k<m\\ k+m\neq 0}} e_{kk}\otimes e_{mm}+\sum_{\substack{k\neq m\\ k+m\neq 0}} e_{km}\otimes e_{mk}+\sum_{k,m} a_{km} e_{k-m}\otimes e_{-km} \tag{2.11}$$

where

$$u_k=q \qquad \text{or} \qquad -q^{-1} \qquad \text{for } k\neq 0$$

and

$$u_{-k}=u_k.$$

The a_{km} are given by

$$a_{km}=\begin{cases} 1 & (k=m=0)\\ u_k^{-1} & (k=m\neq 0)\\ w\left[1-u_m^{-1}\left(\prod_{j=1}^{m-1} u_j^{-2}\right)\right] & (k=-m<0)\\ (-1)^{k+m+1} w u_{k+m}^{-1/2} \prod_{j=1}^{|k+m|-1} u_j^{-1} & (k=0<m \text{ or } k<m=0)\\ (-1)^{k+m+1} w u_m^{-1/2} u_k^{-1/2}\left(\prod_{j=|k|+1}^{|m|-1} u_j^{-1}\right) & (0<k<m \text{ or } k<m<0)\\ (-1)^{k+m+1} w u_m^{-1/2} u_k^{-1/2}\left(\prod_{j=|k|}^{m-1} u_j^{-1}\right)\left(\prod_{i=1}^{|k|-1} u_i^{-2}\right) & (k<0,\ m>0,\ k+m\neq 0) \end{cases} \tag{2.12}$$

for $B_n^{(1)}$

$$a_{km}=\begin{cases} u_k^{-1} & (k=m)\\ w\left[1-\varepsilon u_m^{-1}\left(\prod_{j=1}^{m-1/2} u_{j-1/2}^{-2}\right)u_{1/2}\right] & (k=-m<0)\\ -w u_m^{-1/2} u_k^{-1/2}\left(\prod_{j=1}^{m-1/2} u_{j-1/2}^{-1}\right)u_k^{-1/2} & (0<k<m \text{ or } k<m<0)\\ -\varepsilon w u_m^{-1/2}\left(\prod_{j=|k|+1/2}^{m-1/2} u_{j-1/2}^{-1}\right)\left(\prod_{i=1}^{|k|+1/2} u_{i-1/2}^{-2}\right)u_k^{1/2}u_{1/2}^{\varepsilon} & (k<0,\ m>0,\ k+m\neq 0) \end{cases} \tag{2.13}$$

where $-\varepsilon=1$ for $C_n^{(1)}$ and $\varepsilon=1$ for $D_n^{(1)}$.
The distinct eigenvalues are given by

$$(T-\lambda_1)(T-\lambda_2)(T-\lambda_3)=0 \tag{2.14}$$

where

$$\begin{pmatrix} & \lambda_1 & \lambda_2 & \lambda_3\\ B_n & q & -q^{-1} & \left(\prod_{j=1}^{n} u_j^{-2}\right)\\ C_n & -q^{-1} & q & -\left(\prod_{j=1}^{n} u_{j-1/2}^{-2}\right)u_{1/2}^{-1}\\ D_n & q & -q^{-1} & \left(\prod_{j=1}^{n} u_{j-1/2}^{-2}\right)u_{1/2} \end{pmatrix}. \tag{2.15}$$

When one of u_is is not equal to q the eigenvalues are different from the standard ones, i.e. we meet new solutions It is easy to understand that the solutions derived are natural generalizations of the standard ones from the point of view of the block-diagonal matrix structures of BGRs [14, 19]. Based on the general discussion the BGR under consideration possesses the form

$$T=\text{block diag}(T_1, T_2, \ldots, T_{n-1}, T_n, T_{n-1}, \ldots, T_2, T_1) \tag{2.16}$$

where T_n is an $n \times n$ submatrix For each odd-dimensional submatrix there is a 'central element' denoted by u_{2m+1}. The parameters appearing in the sub-blocks will be determined by the substitution of one sub-block by another in the braid relations. The first one is simply q and the second one should be

$$\begin{bmatrix} 0 & q \\ q & w \end{bmatrix}$$

However, there is more than one possibility for the third one in solving the braid relations. It allows the central element to be either q or $-q^{-1}$, which gives rise to quite different forms of the other sub-blocks by braid relations. When all of the central elements are q our solutions are the same as Jimbo's [7]. However, if one of the central elements is equal to $-q^{-1}$ we will meet new solutions of the BGR. It is not difficult to check that the results in [20] are special cases of our general forms.

3. Birman–Wenzl algebraic properties

It has been known that the standard solutions obey Birman–Wenzl (BW) algebra [10]. Now let us show that the new solutions (2.11)–(2.15) still satisfy the BW algebra. It is pointed out that a BGR T and the related $E = I - w^{-1}(T - T^{-1})(w = q - q^{-1})$ satisfy the BW algebra if they satisfy [22]

(i) $$E^{ab}_{cd} = r(a)r(c)\delta(a, b')\delta(c, d') \tag{3.1}$$

(ii) $$r(a)r(a') = \pm 1 \qquad (a' = N + 1 - a) \tag{3.2}$$

(iii) $$\sum_b T^{ab}_{ab} r^2(b) = \lambda_3^{-1} \tag{3.3}$$

and

$$T^{ab}_{cd} \neq 0 \qquad \text{only for } a + b = c + d \tag{3.4}$$

$$T^{ab}_{cd} = T^{b'a'}_{d'c'} \tag{3.5}$$

with $a' = N + 1 - a$, a belongs to (2.5) $\delta(a, c) = 1$ for $a = c$ and $\delta(a, c) = 0$ for $a \neq c$. Under such a convention and the labelling set L as well as the notation

$$(e_{ab})_{ij} = \delta(a, i) \cdot \delta(b, j) \tag{3.6}$$

the BGR T can be recast in the form

$$T = \sum_{i \neq i'} u_i e_{ii} \otimes e_{ii} + \sum_{\substack{i \neq j, j' \\ \text{or } i = j = j'}} e_{ij} \otimes e_{ji} + \sum_{i \neq i'} u_i^{-1} e_{ii'} \otimes e_{i'i}$$
$$+ w \sum_{i < j} e_{ii} \otimes e_{jj} - w \sum_{i < j} r(i) r(j') e_{j'i} \otimes e_{ji} \tag{3.7}$$

where

$$u_{i'} = u_i \qquad u_i = q \text{ or } -q^{-1} \tag{3.8}$$

$$u_0 = 1 \qquad (\text{only for } B_n) \tag{3.9}$$

and under (2.9)

$$r(i)=\begin{cases}(-1)^i u_i^{-1/2}\prod_{j=0}^{i} u_j & i\geqslant 0 \text{ for } B_n\\ i u_{1/2}^{-1/2}u_i^{1/2}\prod_{j=1/2}^{i} u_j & i>\frac{1}{2} \text{ for } C_n\\ u_{1/2}^{-1/2}u_i^{-1/2}\prod_{j=1/2}^{i} u_j & i>\frac{1}{2} \text{ for } D_n\end{cases} \tag{3.10}$$

Noting that

$$r(i)=r^{-1}(-i) \qquad (i<0) \tag{3.11}$$

in terms of (3 11) it is not difficult to prove that the BGR T found in section 2 satisfies (3.1)-(3 5), i.e. obeys BW algebra. The operator E is defined by (3.1) where $r(i)$ is given by (3.10). The direct check shows that T and E satisfy the BW algebra. Introducing

$$T_j=I\otimes I\otimes\ldots\otimes T\otimes . \quad \otimes I \tag{3.12}$$

we have

$$\begin{aligned}&T_jT_{j\pm1}T_j=T_{j\pm1}T_jT_{j\pm1} \qquad T_jT_k=T_kT_j \qquad |j-k|\geqslant 2\\ &T_j^{-1}-T_j^{-1}=w(I-E_j)\\ &E_jE_{j\pm1}E_j=E_j \qquad E_jE_k=E_kE_j \qquad |j-k|\geqslant 2\\ &E_jT_j=T_jE_j=E_j \qquad T_{j\pm1}T_jE_{j\pm1}=E_jT_{j\pm1}T_j=E_jE_{j\pm1}\\ &T_{j\pm1}E_jT_{j\pm1}=T^{-1}E_{j\pm1}T_j^{-1}\\ &E_{j\pm1}E_jT_{j\pm1}=E_{j\pm1}T_j^{-1} \qquad T_{j\pm1}E_jE_{j\pm1}=T_j^{-1}E_{j\pm1}\\ &E_jT_{j\pm1}E_j=\lambda^{-1}E_j \qquad E_j^2=\left(1-\frac{\lambda-\lambda^{-1}}{w}\right)E_j\end{aligned} \tag{3.13}$$

where $\lambda=\lambda_3$ and $w=q-q^{-1}$.

We emphasize that by interchanging λ_1 and λ_2 and leaving λ_3 unchanged we still obtain the BW algebra. This point is important to generate the solutions of YBE associated with $A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$ from the BGRs associated with $B_n^{(1)}$ and $D_n^{(1)}$.

4. Yang–Baxterization

Since the general solutions (2.11)-(2.13) satisfy the BW algebra it is easy to perform the Yang-Baxterization according to the discussions in [15, 21, 22]. As was pointed out in [15] and [21] there are two ways to Yang-Baxterize the BW algebra. One solution of the YBE is

$$\check{R}_a(x)=\lambda_1x(x-1)T^{-1}+(1+\lambda_1/\lambda_2+\lambda_2/\lambda_3+\lambda_1/\lambda_3)xI-\lambda_3^{-1}(x-1)T \tag{4.1}$$

and the other one, denoted by $\check{R}_b(x)$, can be obtained by $\lambda_1\leftrightarrow\lambda_2$ and $\lambda_3\leftrightarrow\lambda_3$ in $\check{R}_a(x)$. For simplicity we only write the explicit form for case (a) under (2.9):

$$\begin{aligned}\check{R}_a(x)=&\sum_{k\neq0}u_ke_{kk}\otimes e_{kk}-(q^2-1)(x-\xi)\Bigg(\sum_{\substack{k<m\\k+m\neq0}}+x\sum_{\substack{k>m\\k+m\neq0}}\Bigg)e_{kk}\otimes e_{mm}\\ &+q(x-1)(x-\xi)\sum_{\substack{k\neq m\\k+m\neq0}}e_{km}\otimes e_{mk}+\sum_{k,m}a_{km}(x)e_{k-m}\otimes e_{-km}\end{aligned} \tag{4.2}$$

where

$$u_k(x)=\begin{cases}(x-q^2)(x-\xi) & (u_k=q)\\ -q^2(x-q^{-2})(x-\xi) & (u_k=-q^{-1})\end{cases} \tag{4.3}$$

and

$$a_{km}(x)=q(x-1)(x\tilde{a}_{km}-\xi a_{km})+(\xi-1)(q^2-1)x\delta_{k-m}$$

$$\xi=\begin{cases}q^{-1}\lambda_3^{-1} & \text{for } B_n^{(1)} \text{ and } D_n^{(1)}\\ -q\lambda_3^{-1} & \text{for } C_n^{(1)}\end{cases} \tag{4.4}$$

$$\tilde{a}_{km}(u_k)=a_{mk}(u_k^{-1}). \tag{4.5}$$

The $\check{R}_b(x)$ can be obtained by the formal interchange $q\leftrightarrow -q^{-1}$ and keeping λ_3 unchanged. The solutions thus derived correspond to the 'twisted' ones. For instance, corresponding to $\check{R}_a$s of $B_n^{(1)}$ and $D_n^{(1)}$, case (b) gives the solutions relating to $A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$, respectively. This is easy to check by means of the standard solutions. We note that case (b) is the 'normal' solution for C_n; however, case (a) is still a solution We do not know about the Lie algebraic description of case (b) for C_n yet. All of the new 'twisted' solutions deserve to be understood.

It is worth discussing the eigenvalues in detail, since they relate to the diagonalization of BGRs. The first two eigenvalues $\lambda_1=u=q$ and $\lambda_2=v=-q^{-1}$ appear only in the sub-blocks other than the largest one, whereas the third eigenvalue $\lambda=\lambda_3$ only appears in the largest sub-blocks. We thus focus on considering the eigenvalue equation of the largest sub-blocks. By calculation we find that

$$(\lambda-u)^{2n-1-n_1}(\lambda-v)^{n_1}\left[\lambda+\left(\prod_{j=1}^{n}u_{j-1/2}^{-2}\right)u_{1/2}^{-1}\right]=0$$

when

$$\left(\prod_{j=1}^{n}u_{j-1/2}^{-2}\right)u_{1/2}^{-1}\neq\pm u \text{ or } \pm v \qquad \text{for } C_n(n_1=n \text{ or } n-1)$$

and

$$(\lambda-u)^{n-1-n_1}(\lambda-v)^{n_1}\left[\lambda-\left(\prod_{j=1}^{n}u_{j-1/2}^{-2}\right)u_{1/2}\right]=0$$

when

$$\left(\prod_{j=1}^{n}u_{j-1/2}^{-2}\right)u_{1/2}\neq\pm u \text{ or } \pm v \qquad \text{for } D_n(n_1=n \text{ or } n-1).$$

There are three distinct eigenvalues for B_n, whereas when

$$\left(\prod_{j=1}^{n}u_{j-1/2}^{-2}\right)u_{1/2}^{-1} \qquad \text{for } C_n\left(\text{or}\left(\prod_{j=1}^{n}u_{j-1/2}^{-2}\right)u_{1/2} \text{ for } D_n\right)$$

is equal to $\pm u$ or $\pm v$ the eigenvalue equation is reduced to

$$(\lambda-u)^n(\lambda-v)^n=0$$

namely, the number of distinct eigenvalues is no longer the same as the decomposition dimension, being two rather than three. For instance, for C_2 we have the minimum polynomial

$$(T-u)^2(T-v)=0.$$

The multiplicity of two in (4.6) is essential This indicates that the corresponding BGR cannot be diagonalized. One can check this statement generally or explicitly through example, say C_2 Even in such a case it is proved that solutions of BGRs can still be Yang–Baxterized by either case (a) or case (b).

5. Quantum group structure

As is known, for a given BGR the corresponding quantum group can be constructed with the help of the Faddeev–Reshetikhin–Takhtajan (FRT) approach [9]. However, for our problem the direct application of FRT is complicated and we, therefore, follow [11], using an equivalent but simple technique to set up desired relations satisfied by the generators of quantum groups (i.e. algebra). Suppose

$$\check{R}(x) = A(x)T + B(x)I + C(x)T^{-1} \tag{5.1}$$

satisfies the YBE

$$\check{R}_{12}(x)\check{R}_{23}(xy)\check{R}_{12}(y) = \check{R}_{23}(y)\check{R}_{12}(xy)\check{R}_{23}(x) \tag{5.2}$$

then the Yang–Baxter operator can be defined by [11]

$$(t_{ab}(x))_{cd} = \check{R}^{ca}_{bd}(x) \tag{5.3}$$

in accordance with

$$\check{R}(xy^{-1})(t(x)\otimes t(y)) = (t(y)\otimes t(x))\check{R}(xy^{-1}). \tag{5.4}$$

Following [11] the asymptotic behaviour of $t(x)_{x\to 0}$ and $t(x)_{x^{-1}\to 0}$ gives the generators of quantum group by omitting the factors $A(x)_{x\to 0}$ and $C(x)_{x^{-1}\to 0}$ (even the limits tend to infinity).

The calculations are lengthy so we only give the results.

(i) For B_n we have

$$
\begin{aligned}
&(t_{ii}(x))_{ab}|_{x\to 0}\\
&\quad = A(x)|_{x\to 0}\{u_i\delta_{ab}\delta_{ai} + \delta_{ab}|_{\substack{a\neq i,i'\\ a=n+1}} + u_i^{-1}\delta_{ab}\delta_{i'a}\}\\
&\quad = A(x)|_{x\to 0}(k_i)_{ab}\\
&(t_{ii}(x))_{ab}|_{x\to\infty}\\
&\quad = C(x)|_{x\to\infty}\{u_i^{-1}\delta_{ab}\delta_{ai} + \delta_{ab}|_{\substack{a\neq i,i'\\ a=n+1}} + u_i\delta_{ab}\delta_{i'a}\}\\
&\quad = C(x)|_{x\to\infty}(k_i^{-1})_{ab}\\
&(t_{i+1\,i}(x))_{ab}|_{x\to 0}\\
&\quad = A(x)|_{x\to 0}\{w\delta_{ai}\delta_{i+1b} - wr(i)r(N-i)\delta_{N-ia}\delta_{i'b}\}\\
&\quad = A(x)|_{x\to 0}w(e_i)_{ab}\\
&(t_{i\,i+1}(x))_{ab})|_{x\to\infty}\\
&\quad = C(x)|_{x\to\infty}\{-w\delta_{ai+1}\delta_{ib} + wr(i+1)r(i')\delta_{i'a}\delta_{N-ib}\}\\
&\quad = C(x)|_{x\to\infty}(-w)(f_i)_{ab}.
\end{aligned}
$$

Introducing

$$K_i = k_i k_{i+1}^{-1} \qquad K_n = k_n^2$$
$$X_i^+ = e_i \qquad X_i^- = f_i \qquad i = 1, 2, \ldots, n-1$$
$$X_n^+ = (u_n + u_n^{-1})^{1/2} e_n \qquad X_n^- = (u_n + u_n^{-1})^{1/2} f_n$$

we obtain

$$[X_i^+, X_j^-] = \delta_{ij}(K_i - K_i^{-1})/w \qquad i, j = 1, \ldots, n$$
$$K_i X_i^{\pm} K_i^{-1} = (u_i u_{i+1})^{\pm 1} X_i \qquad i = 1, \ldots, n-1$$
$$K_{i-1} X_i^{\pm} K_{i-1}^{-1} = u_i^{\mp 1} X_i^{\pm} \qquad i = 1, \ldots, n$$
$$K_{i+1} X_i^{\pm} K_{i+1}^{-1} = u_{i+1}^{\mp 1} X_i \qquad i = 1, \ldots, n-2$$
$$K_n X_n^{\pm} K_n^{-1} = (u_n)^{\pm 2} X_n^{\pm} \qquad K_n X_{n-1}^{\pm} K_n^{-1} = (u_n)^{\mp 2} X_n$$
$$K_j X_i^{\pm} K_j^{-1} = X_i \qquad |i-j| > 1,\ K_i K_j = K_j K_i$$

The Serre relations become, under the representation,

$$(X_i^{\pm})^2 = 0 \qquad i = 1, \ldots, n-1 \qquad (X_n^{\pm})^3 = 0$$

The coproducts read

$$\Delta(X_i^+) = k_{i+1} \otimes X_i^+ + X_i^+ \otimes k_i \qquad i = 1, \ldots, n-1$$
$$\Delta(X_n^+) = I \otimes X_n^+ + X_n^+ \otimes k_n$$
$$\Delta(X_i^-) = k_i^{-1} \otimes X_i^- + X_i^- \otimes k_{i+1}^{-1}$$
$$\Delta(X_n^-) = k_n^{-1} \otimes X_n^- + X_n^- \otimes I \qquad \Delta(k_i^{\pm}) = k_i^{\pm} \otimes k_i^{\pm}$$

where I stands for the unit matrix.

The antipode and the co-unit are given by

$$\gamma(k_i) = k_i^{-1} \qquad \gamma(I) = I \qquad \varepsilon(X_i^{\pm}) = 0 \qquad \varepsilon(k_i^{\pm 1}) = 1$$
$$\gamma(X_i^+) = -k_{i+1}^{-1} X_i^+ k_i^{-1} \qquad \gamma(X_i^-) = -k_i X_i^- k_{i+1} \qquad i = 1, \ldots, n-1$$
$$\gamma(X_n^+) = -X_n^+ k_n^{-1} \qquad \gamma(X_n^-) = -k_n X_n^-$$

(ii) For C_n, by introducing

$$K_i = k_i k_{i+1}^{-1} \qquad K_n = k_n$$
$$X_i^+ = e_i \qquad X_i^- = f_i \qquad f = 1, \ldots, n-1$$
$$X_n^+ = (1 - r^2(n))^{-1} e_n \qquad X_n^- = (1 - r^2(n+1))^{-1} f_n$$

we derive

$$[X_i^+, X_j^-] = \delta_{ij}(K_i - K_i^{-1})/w \qquad i, j = 1, \ldots, n$$
$$K_i X_i^{\pm} K_i^{-1} = (u_i u_{i+1})^{\pm 1} X_i^{\pm} \qquad i = 1, \ldots, n-1$$
$$K_n X_n^{\pm} K_n^{-1} = (u_n)^{\pm 2} X_n^{\pm}$$
$$K_{i-1} X_i^{\pm} K_{i-1}^{-1} = u_i^{\mp 1} X_i^{\pm} \qquad K_{i-1} X_i^{\pm} K_{i+1}^{-1} = u_{i+1}^{\mp} X_i^{\pm} \qquad i = 1, \ldots, n-1$$
$$K_{n-1} X_n^{\pm} K_{n-1}^{-1} = (u_n)^{\mp 2} X_n^{\mp}$$
$$K_i X_j^{\pm} K_i^{-1} = X_j^{\pm} \qquad |i-j| > 1\,. \qquad K_i K_j = K_j K_i.$$

The Serre relations under these particular representations read

$$(X_i^{\pm})^2=0 \qquad i=1,\ldots,n.$$

The coproducts are given by

$$\left.\begin{aligned}\Delta(X_i^+)&=k_{i+1}\otimes X_i^+ + X_i^+\otimes k_i\\ \Delta(X_i^-)&=k_i^{-1}\otimes X_i^+ + X_i^-\otimes k_{i+1}^{-1}\end{aligned}\right\} \quad i=1,\ldots,n-1$$

$$\Delta(X_n^+)=k_n\otimes X_n^+ + X_n^+\otimes k_n$$

$$\Delta(X_n^-)=k_n^{-1}\otimes X_n^- + X_n^-\otimes k_n^{-1}$$

$$\Delta(k_i^{\pm})=k_i^{\pm}\otimes k_i^{\pm}.$$

The antipode and co-unit are given by

$$\gamma(k_i)=k_i^{-1} \qquad \varepsilon(X_i^{\pm})=0 \qquad \varepsilon(k_i^{\pm})=1$$

$$\gamma(X_i^+)=-k_{i+1}^{-1}X_i^+k_i^{-1} \qquad \gamma(X_i^-)=-k_iX_i^-k_{i+1}$$

$$\gamma(X_n^+)=-k_n^{-1}X_n^+k_n^{-1} \qquad \gamma(X_n^-)=-k_nX_n^-k_n.$$

(iii) For D_n we have

$$\begin{aligned}(t_{n-1\,n+1}(x))_{ab}|_{x\to\infty} &= C(x)|_{x\to\infty}\{-w\delta_{aN+1}\delta_{bn-1}+w\Gamma(n+1)\Gamma(n+2)\delta_{an+2}\delta_{bn}\}\\ &= -C(x)|_{x\to\infty}w(f_n')_{ab}\end{aligned}$$

$$\begin{aligned}(t_{n+1\,n-1}(x))_{ab}|_{x\to0} &= A(x)|_{x\to0}\{w\delta_{an-1}\delta_{bn+1}-w\Gamma(n-1)\Gamma(n)\delta_{an}\delta_{bn+2}\}\\ &= A(x)|_{x\to0}w(e_n')_{ab}.\end{aligned}$$

Introducing

$$K_i=k_ik_{i+1}^{-1} \qquad i=1,\ldots,n-1,\ K_n=k_{n-1}k_n$$

$$X_i^+=e_i \qquad X_i^-=f_i \qquad i=1,\ldots,n-1 \qquad X_n^+=e_n',\ X_n^-=f_n'$$

we obtain the following relations:

$$[X_i^+,X_j^-]=\delta_{ij}(K_i-K_i^{-1})/w \qquad i,j=1,\ldots,n$$

$$K_iX_i^{\pm}K_i^{-1}=(u_iu_{i+1})^{\pm1}X_i^{\pm} \qquad i=1,\ldots,n-1$$

$$K_nX_n^{\pm}K_n^{-1}=(u_{n-1}u_n)^{\pm1}X_n^{\pm}$$

$$K_{i-1}X_i^{\pm}K_{i-1}^{-1}=u_i^{\mp1}X_i^{\pm} \qquad i=1,\ldots,n-1$$

$$K_{i+1}X_i^{\pm}K_{i+1}^{-1}=(u_{i+1})^{\mp1}X_i^{\pm}$$

$$K_{n-1}X_n^{\pm}K_{n-1}^{-1}=u_{n-1}^{\pm1}u_n^{\mp1}X_n^{\pm}$$

$$K_nX_{n-1}^{\pm}K_n^{-1}=(u_{n-1})^{\pm1}u_n^{\mp1}X_{n-1}^{\pm}$$

$$K_nX_{n-2}^{\pm}K_n^{-1}=(u_{n-1})^{\mp1}X_{n-2}^{\pm}$$

$$K_{n-2}X_n^{\pm}K_{n-2}^{-1}=(u_{n-1})^{\mp1}X_n^{\pm}$$

$$K_iX_j^{\pm}K_i^{-1}=X_j^{\pm} \qquad |i-j|>1 \qquad i,j\neq n,n-2$$

$$K_iK_j=K_jK_i \qquad i=1,\ldots,n.$$

Under the chosen representations the Serre relations are

$$(X_i^{\pm})^2=0 \qquad i=1,\ldots,n.$$

The coproducts are given by

$$\left.\begin{aligned}\Delta(X_i^+)&=k_{i+1}\otimes X_i^+ + X_i^+\otimes k_i\\ \Delta(X_i^-)&=k_i^{-1}\otimes X_i^- + X_i^-\otimes k_{i+1}^{-1}\end{aligned}\right\} \quad i=1,\ldots,n-1$$

$$\Delta(X_n^+)=k_n\otimes X_n^+ + X_n^+\otimes k_{n-1}$$

$$\Delta(X_n^-)=k_{n-1}^{-1}\otimes X_n^- + X_n^-\otimes k_n^{-1} \qquad \Delta(k^{\pm1})=k_i^{\pm1}\otimes k_i^{\pm1}.$$

The antipode and co-unit are given by

$$\gamma(k_i)=k_i^- \qquad \varepsilon(X_i^{\mp})=0 \qquad \varepsilon(k_i^{\pm1})=1$$

$$\gamma(X_i^+)=-k_{i+1}^{-1}X_i^+k_i^{-1}$$

$$\gamma(X_i^-)=-k_iX_i^-k_{i+1} \qquad i=1,\ldots,n-1$$

$$\gamma(X_n^+)=-k_n^{-1}X_n^+k_{n-1}^{-1} \qquad \gamma(X_n^-)=-k_{n-1}X_n^-k_n.$$

We would like to emphasize that in the standard cases were $u_i=q$ for all indices i the quantum group derived in this section is nothing but the usual quantum universal enveloping algebra shown by

$$[X_i^+,X_j^-]=\delta_{ij}(K_i-K_i^{-1})/w$$

$$K_iX_jK_i^{-1}=q^{a}ijX_j \qquad K_iK_j=K_jK_i$$

with a_{ij} being the Cartan matrix element of Lie algebras B_n, C_n and D_n.

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